THE SECOND JOHNSON HOMOMORPHISM AND THE SECOND RATIONAL COHOMOLOGY OF THE JOHNSON KERNEL

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ABSTRACT. The Johnson kernel is the subgroup of the mapping class group of a surface generated by Dehn twists along bounding simple closed curves, and has the second Johnson homomorphism as a free abelian quotient. In terms of the representation theory of the symplectic group, we give a complete description of cup products of two classes in the first rational cohomology of the Johnson kernel obtained by the rational dual of the second Johnson homomorphism.

1. INTRODUCTION

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let $\mathcal{M}_g$ be the mapping class group of Σ_g, which is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g. The Torelli group $\mathcal{I}_g$ is the subgroup of $\mathcal{M}_g$ consisting of all elements which act trivially on the first homology group $H := H_1(\Sigma_g)$ of Σ_g, and the group $\mathcal{K}_g$ is the subgroup of $\mathcal{I}_g$ generated by Dehn twists along bounding simple closed curves. Johnson [12] showed that $\mathcal{K}_g$ coincides with the kernel of what is now called the first Johnson homomorphism $\tau_g(1)$ of $\mathcal{I}_g$ (see [10]), namely we have an exact sequence

$$1 \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{I}_g \xrightarrow{\tau_g(1)} \mathfrak{h}_g(1) = (\wedge^3 H)/H \longrightarrow 1,$$

and by this, $\mathcal{K}_g$ is called the *Johnson kernel* or the *Johnson subgroup*.

The group $\mathcal{K}_g$ plays an important role in topology. For example, it has some relationships to the Casson invariant of homology 3-spheres and secondary characteristic classes of surface bundles as we see in Morita's papers [18, 21]. However, we still do not have enough information on $\mathcal{K}_g$. McCullough-Miller [16] showed that $\mathcal{K}_2 = \mathcal{I}_2$ is not finitely generated, and Mess [17] showed that it is a free group of infinite rank. Recently, Biss-Farb [5] showed that $\mathcal{K}_g$ is not finitely generated for all $g \geq 2$. Note that the determination of the abelianization of $\mathcal{K}_g$ is still open.

Both of the roles of $\mathcal{K}_g$ mentioned above can be interpreted as properties of some elements in the rational cohomology $H^*(\mathcal{K}_g;\mathbb{Q})$ of $\mathcal{K}_g$, to which we now pay our attention. By the fact that $\mathcal{K}_g$ is torsion-free and acts on the Teichmüller space properly discontinuously, we can see that $\mathcal{K}_g$ has finite cohomological dimension. On the other hand, Akita [1] showed that $H^*(\mathcal{K}_g;\mathbb{Q})$ is an infinite-dimensional vector space.

Our strategy to study $H^*(\mathcal{K}_g;\mathbb{Q})$ is to use the second Johnson homomorphism

$$\tau_g(2) : \mathcal{K}_g \longrightarrow \mathfrak{h}_g(2),$$

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where $\mathfrak{h}_g(2)$ is a certain free abelian group which can be described by using H (see Section 2 for the notation used here). An important fact of $\tau_g(2)$ is that it is $\mathcal{M}_g$-equivariant, where $\mathcal{M}_g$ acts on $\mathcal{K}_g$ by conjugation and acts on $\mathfrak{h}_g(2)$ through its action on H known as the classical representation $\mathcal{M}_g \to Sp(2g,\mathbb{Z})$. Moreover, if we consider the induced map

$$\tau_g(2)_* : H_1(\mathcal{K}_g;\mathbb{Q}) \longrightarrow H_1(\mathfrak{h}_g(2);\mathbb{Q}) = \mathfrak{h}_g(2) \otimes \mathbb{Q},$$

we can check that its image is preserved by the action of $Sp(2g,\mathbb{Q})$ extending that of $Sp(2g,\mathbb{Z})$ (see Asada-Nakamura cite[Lemma 2.2.8]an). In terms of the symplectic representation theory, $\mathfrak{h}_g(2) \otimes \mathbb{Q}$ is the irreducible representation denoted by $[2^2]$ (see [9], [18]). By passing to the dual, we obtain an injection

$$\tau_g(2)^* : [2^2] \longrightarrow H^1(\mathcal{K}_g;\mathbb{Q}),$$

and more generally, we have the cup product map

$$\cup^n : \wedge^n[2^2] \longrightarrow H^n(\mathcal{K}_g;\mathbb{Q}).$$

Note that $\wedge^n[2^2]$ and $\mathrm{Ker}\,\cup^n$ are also $Sp(2g,\mathbb{Q})$-vector spaces for each $n \geq 2$. In this paper, we study the case of $n = 2$. We put $\cup := \cup^2$, for simplicity. We will show the following.

Lemma 4.1. *For $g \geq 4$, the irreducible decomposition of $\wedge^2[2^2]$ is given by*

$$\wedge^2[2^2] = [431] + [42] + [32^21] + [321] + [31^3] + [31] + [2^3] + [21^2] + [2].$$

Theorem 5.1. *For $g \geq 4$, the kernel of the cup product map $\cup : \wedge^2[2^2] \to H^2(\mathcal{K}_g;\mathbb{Q})$ is*

$$[42] + [31^3] + [31] + [2^3] + [2],$$

which is, as an $Sp(2g,\mathbb{Q})$-vector space, isomorphic to the rational image of the fourth Johnson homomorphism $\tau_g(4)$.

Here we write $+$ for the direct sum. Note that our theorem can be stated that we determine the kernel of the map induced on the second rational cohomology by $\tau_g(2)$. We will also treat the cases of $g = 2, 3$ in Section 6.

Before proving Theorem 5.1, which corresponds to the case of a closed surface, we show a similar result for the case of a surface with a boundary in Section 4, since it is easier to handle by a technical reason. In each case, our task is divided into the following two parts. First, we will find some summands in Lemma 4.1 (or that corresponding to each case) which belong to the kernel by using Stallings' exact sequence in [25] together with Morita's description [19, 20] of Johnson's homomorphisms as a Lie algebra homomorphism. Then we show that the other summands actually survive in the second cohomology by constructing explicit cycles which come from abelian subgroups of the Johnson kernel and give non-trivial values by the Kronecker product.

Note that the method we have explained now originated with Hain [9], where he determined the kernel of the map induced on the second rational cohomology by the first Johnson homomorphism for the Torelli group. Our previous paper [24] treated the third cohomology of the Torelli group. Brendle-Farb [6] studied the second cohomology of the Torelli group and the Johnson kernel by using the Birman-Craggs-Johnson homomorphism, and Pettet [23] studied

the second cohomology of the (outer-)automorphism group of a free group by using its first Johnson homomorphism.

2. Preliminaries

2.1. **Surfaces and their mapping class groups.** Let $H_1(\Sigma_g)$ be the first integral homology group of a closed oriented surface Σ_g of genus $g \geq 2$. $H_1(\Sigma_g)$ has a natural intersection form $\mu : H_1(\Sigma_g) \otimes H_1(\Sigma_g) \to \mathbb{Z}$ which is non-degenerate and skew symmetric. We fix a symplectic basis $\langle a_1, \dots, a_g, b_1, \dots, b_g \rangle$ of $H_1(\Sigma_g)$ with respect to μ, namely

$$\mu(a_i, a_j) = 0, \quad \mu(b_i, b_j) = 0, \quad \mu(a_i, b_j) = \delta_{ij}.$$

The Poincaré duality gives a canonical isomorphism of $H_1(\Sigma_g)$ with its dual $H_1(\Sigma_g)^* = H^1(\Sigma_g)$, the first integral cohomology group of Σ_g. In this isomorphism, a_i (resp. b_i) $\in H_1(\Sigma_g)$ corresponds to $-b_i^*$ (resp. a_i^*) $\in H^1(\Sigma_g)$ where $\langle a_1^*, \dots, a_g^*, b_1^*, \dots, b_g^* \rangle$ is the dual basis of $H^1(\Sigma_g)$. We use the same symbol H for these canonically isomorphic abelian groups.

We also use a compact oriented surface $\Sigma_{g,1}$ of genus g with a connected boundary. $H_1(\Sigma_{g,1})$ can be naturally identified with H. The fundamental group $\pi_1\Sigma_{g,1}$ of $\Sigma_{g,1}$, where we take a base point of $\Sigma_{g,1}$ on $\partial\Sigma_{g,1}$, is known to be a free group of rank $2g$. We write $\zeta \in \pi_1\Sigma_{g,1}$ for the boundary loop of $\Sigma_{g,1}$. Then the fundamental group $\pi_1\Sigma_g$ of Σ_g is given by $\pi_1\Sigma_{g,1}/\langle\zeta\rangle$ where $\langle\zeta\rangle$ is the normal closure of the subgroup generated by ζ.

Let $\mathcal{M}_g, \mathcal{M}_{g,*}, \mathcal{M}_{g,1}$ be the mapping class group of Σ_g, of Σ_g relative to the base point, of $\Sigma_{g,1}$, respectively. They are related by the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{M}_{g,1} & \longrightarrow & \mathcal{M}_{g,*} & \longrightarrow & 1, \\ 1 & \longrightarrow & \pi_1\Sigma_g & \longrightarrow & \mathcal{M}_{g,*} & \longrightarrow & \mathcal{M}_g & \longrightarrow & 1, \end{array}$$

where $\mathbb{Z}$ corresponds to the Dehn twist along a loop which is parallel to $\partial\Sigma_{g,1}$, and $\pi_1\Sigma_g$ is embedded in $\mathcal{M}_{g,*}$ as spin-maps (see [4, Theorem 4.3]). The former sequence is a central extension.

The natural action of $\mathcal{M}_g$ on H gives the classical representation

$$\mathcal{M}_g \longrightarrow Sp(2g, \mathbb{Z}),$$

and we also have similar ones for $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,1}$. The kernels of these representations are denoted by $\mathcal{I}_g$, $\mathcal{I}_{g,*}$ and $\mathcal{I}_{g,1}$, respectively and called the *Torelli group* for each case. Note that among $\mathcal{I}_g$, $\mathcal{I}_{g,*}$ and $\mathcal{I}_{g,1}$, we also have exact sequences similar to the above. Indeed the Dehn twist along $\partial\Sigma_{g,1}$ and spin-maps act on H trivially.

Let $\mathcal{K}_g$ (resp. $\mathcal{K}_{g,1}$) be the subgroup of $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) generated by Dehn twists along bounding simple closed curves on Σ_g (resp. $\Sigma_{g,1}$). We define $\mathcal{K}_{g,*} \subset \mathcal{M}_{g,*}$ to be the image of $\mathcal{K}_{g,1}$ by the map $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$. Then we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{K}_{g,1} & \longrightarrow & \mathcal{K}_{g,*} & \longrightarrow & 1, \\ 1 & \longrightarrow & [\pi_1\Sigma_g, \pi_1\Sigma_g] & \longrightarrow & \mathcal{K}_{g,*} & \longrightarrow & \mathcal{K}_g & \longrightarrow & 1, \end{array}$$

where the former sequence is the pull-back of the central extension of $\mathcal{M}_{g,*}$, and the latter one follows from a result of Asada-Kaneko [2].

2.2. **Johnson's homomorphisms.** In this subsection, we recall what we call Johnson's homomorphisms defined by Johnson [10, 11] and Morita [18, 19, 20, 21].

By results of Dehn, Nielsen and many people, we have natural isomorphisms

$$\mathcal{M}_g \cong \mathrm{Out}_+\,\pi_1\Sigma_g, \qquad \mathcal{M}_{g,*} \cong \mathrm{Aut}_+\,\pi_1\Sigma_g, \qquad \mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\,\pi_1\Sigma_{g,1} \mid \varphi(\zeta)=\zeta\},$$

where $\mathrm{Out}_+\,\pi_1\Sigma_g := \mathrm{Ker}(\mathrm{Out}\,\pi_1\Sigma_g \to \mathrm{Aut}\,H_2(\pi_1\Sigma_g))$, and $\mathrm{Aut}_+\,\pi_1\Sigma_g$ is similar.

For a group G, let $\{\Gamma^k G\}_{k\geq 1}$ be the lower central series of G inductively defined by $\Gamma^1 G = G$ and $\Gamma^i G = [\Gamma^{i-1}G, G]$ for $i \geq 2$. By a general theory, $\{(\Gamma^k G)/(\Gamma^{k+1}G)\}_{k\geq 1}$ forms a graded Lie algebra whose bracket map is induced from taking commutators. It is well known that the Lie algebra $\{(\Gamma^k\pi_1\Sigma_{g,1})/(\Gamma^{k+1}\pi_1\Sigma_{g,1})\}_{k\geq 1}$ is isomorphic to the free Lie algebra $\mathcal{L}_{g,1} = \{\mathcal{L}_{g,1}(k)\}_{k\geq 1}$ generated by H. Furthermore, by a result of Labute [13], the Lie algebra $\{(\Gamma^k\pi_1\Sigma_g)/(\Gamma^{k+1}\pi_1\Sigma_g)\}_{k\geq 1}$ is given by $\mathcal{L}_g := \mathcal{L}_{g,1}/I$ where I is the ideal of $\mathcal{L}_{g,1}$ generated by $\omega_0 := \sum_{i=1}^g [a_i, b_i]$.

The isomorphisms mentioned above induce the homomorphisms

$$\begin{aligned}\sigma_k\ :\ \mathcal{M}_g &\longrightarrow \mathrm{Out}\big(\pi_1\Sigma_g/(\Gamma^k\pi_1\Sigma_g)\big),\\ \sigma_{k,*} : \mathcal{M}_{g,*} &\longrightarrow \mathrm{Aut}\big(\pi_1\Sigma_g/(\Gamma^k\pi_1\Sigma_g)\big),\\ \sigma_{k,1} : \mathcal{M}_{g,1} &\longrightarrow \mathrm{Aut}\big(\pi_1\Sigma_{g,1}/(\Gamma^k\pi_1\Sigma_{g,1})\big),\end{aligned}$$

for each $k \geq 2$, and we define filtrations of $\mathcal{M}_g, \mathcal{M}_{g,*}, \mathcal{M}_{g,1}$ by

$$\begin{aligned}\mathcal{M}_g[1] &:= \mathcal{M}_g, & \mathcal{M}_g[k] &:= \mathrm{Ker}\,\sigma_k \quad (k\geq 2),\\ \mathcal{M}_{g,*}[1] &:= \mathcal{M}_{g,*}, & \mathcal{M}_{g,*}[k] &:= \mathrm{Ker}\,\sigma_{k,*} \quad (k\geq 2),\\ \mathcal{M}_{g,1}[1] &:= \mathcal{M}_{g,1}, & \mathcal{M}_{g,1}[k] &:= \mathrm{Ker}\,\sigma_{k,1} \quad (k\geq 2).\end{aligned}$$

For each $\varphi \in \mathcal{M}_{g,1}[k+1]$ and $\gamma \in \pi_1\Sigma_{g,1}$, we have $\varphi(\gamma)\gamma^{-1} \in \Gamma^{k+1}\pi_1\Sigma_{g,1}$. This induces a map $\tau_{g,1}(k) : \mathcal{M}_{g,1}[k+1] \to \mathrm{Hom}(\pi_1\Sigma_{g,1}, (\Gamma^{k+1}\pi_1\Sigma_{g,1})/(\Gamma^{k+2}\pi_1\Sigma_{g,1})) = \mathrm{Hom}(H, \mathcal{L}_{g,1}(k+1))$, and it is in fact a homomorphism.

In [19, 20], Morita showed the following. By taking commutators, we can endow $\{\mathcal{M}_{g,1}[k+1]/\mathcal{M}_{g,1}[k+2]\}_{k\geq 1} = \{\mathrm{Im}\,\tau_{g,1}(k)\}_{k\geq 1} =: \mathrm{Im}\,\tau_{g,1}$ with a Lie algebra structure. We can also endow $\mathrm{Hom}(H,\mathcal{L}_{g,1}) := \{\mathrm{Hom}(H,\mathcal{L}_{g,1}(k+1))\}_{k\geq 1}$ with a Lie algebra structure, so that $\tau_{g,1} := \{\tau_{g,1}(k)\}_{k\geq 1}$ becomes a Lie algebra inclusion of $\mathrm{Im}\,\tau_{g,1}$ into $\mathrm{Hom}(H,\mathcal{L}_{g,1})$. Moreover, $\mathrm{Im}\,\tau_{g,1}$ is contained in the Lie subalgebra $\mathfrak{h}_{g,1} = \{\mathfrak{h}_{g,1}(k)\}_{k\geq 1}$ defined by

$$\mathfrak{h}_{g,1}(k) := \mathrm{Ker}\left(\mathrm{Hom}(H,\mathcal{L}_{g,1}(k+1)) \xrightarrow{\cong} H\otimes\mathcal{L}_{g,1}(k+1) \xrightarrow{[\cdot,\cdot]} \mathcal{L}_{g,1}(k+2)\right),$$

where the maps in the right hand side are given by the Poincaré duality and the bracket operation. A similar argument gives a homomorphism $\tau_{g,*}(k) : \mathcal{M}_{g,*}[k+1] \to \mathfrak{h}_{g,*}(k) \subset \mathrm{Hom}(H,\mathcal{L}_g(k+1))$, where

$$\mathfrak{h}_{g,*}(k) := \mathrm{Ker}\left(\mathrm{Hom}(H,\mathcal{L}_g(k+1)) \xrightarrow{\cong} H\otimes\mathcal{L}_g(k+1) \xrightarrow{[\cdot,\cdot]} \mathcal{L}_g(k+2)\right),$$

and the corresponding Lie algebra inclusion. Furthermore, Asada-Kaneko [2] showed that $\pi_1\Sigma_g \cap \mathcal{M}_{g,*}[k+1] = \Gamma^k\pi_1\Sigma_g$. Hence we have an inclusion

$$\Psi_k : (\Gamma^k\pi_1\Sigma_g)/(\Gamma^{k+1}\pi_1\Sigma_g) \cong \mathcal{L}_g(k) \hookrightarrow \mathfrak{h}_{g,*}(k).$$

If we set $\mathfrak{h}_g(k) := \mathfrak{h}_{g,*}(k)/\mathcal{L}_g(k)$, we obtain a homomorphism $\tau_g(k) : \mathcal{M}_g[k+1] \to \mathfrak{h}_g(k)$ (and the corresponding Lie algebra inclusion). We call the homomorphisms $\tau_g(k)$, $\tau_{g,*}(k)$, $\tau_{g,1}(k)$ the k-th Johnson homomorphism for each case. Note that $\tau_g(k)$ is $\mathcal{M}_g$-equivariant, where $\mathcal{M}_g$ acts on $\mathcal{M}_g[k+1]$ by conjugation and acts on the target through the classical representation $\mathcal{M}_g \to Sp(2g, \mathbb{Z})$. Similar results hold for $\tau_{g,*}(k)$ and $\tau_{g,1}(k)$.

We have $\mathcal{M}_{g,1}[2] = \mathcal{I}_{g,1}$, $\mathcal{M}_{g,*}[2] = \mathcal{I}_{g,*}$ and $\mathcal{M}_g[2] = \mathcal{I}_g$ by definition. Johnson [12] showed that $\mathcal{M}_{g,1}[3] = \mathcal{K}_{g,1}$ and $\mathcal{M}_g[3] = \mathcal{K}_g$. Combining the fact that the first Johnson homomorphisms for $\mathcal{I}_{g,1}$ and $\mathcal{I}_{g,*}$ have the same target $\wedge^3 H$, we can see that $\mathcal{M}_{g,*}[3] = \mathcal{K}_{g,*}$.

2.3. **The representation theory of $Sp(2g, \mathbb{Q})$.** Here we summarize the notation and general facts concerning the representation theory of $Sp(2g, \mathbb{Q})$ from [7], [9] and [21]. First we consider the Lie group $Sp(2g, \mathbb{C})$ and its Lie algebra $\mathfrak{sp}(2g, \mathbb{C})$. By a general theory of the representation, finite dimensional representations of $Sp(2g, \mathbb{C})$ coincide with those of $\mathfrak{sp}(2g, \mathbb{C})$, and their common irreducible representations (up to isomorphisms) are parameterized by Young diagrams whose numbers of rows are less than or equal to g. These representations are all defined over $\mathbb{Q}$ so that we can consider them as irreducible representations of $Sp(2g, \mathbb{Q})$ and $\mathfrak{sp}(2g, \mathbb{Q})$. We follow the notation in [21] to describe Young diagrams. For example, the trivial representation $\mathbb{Q}$ is denoted by $[0]$ and the fundamental representation $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$ is denoted by $[1]$. We fix a symplectic basis $\langle a_1, \dots, a_g, b_1, \dots, b_g \rangle$ of $H_{\mathbb{Q}}$ with respect to the non-degenerate skew symmetric bilinear form, denoted by μ again, on $H_{\mathbb{Q}}$ induced from the intersection form μ on H. In general, the Young diagram $[n_1 n_2 \cdots n_l]$, where n_i are integers satisfying $n_1 \geq n_2 \geq \cdots \geq n_l \geq 1$ and $l \leq g$, corresponds to the $Sp(2g, \mathbb{Q})$-vector space V given as follows. Let $[m_1 m_2 \cdots m_k]$ be the Young diagram obtained by transposing $[n_1 n_2 \cdots n_l]$. Then V is explicitly defined to be the irreducible $Sp(2g, \mathbb{Q})$-subspace of

$$(\wedge^{m_1} H_{\mathbb{Q}}) \otimes (\wedge^{m_2} H_{\mathbb{Q}}) \otimes \cdots \otimes (\wedge^{m_k} H_{\mathbb{Q}})$$

containing the vector

$$(a_1 \wedge a_2 \wedge \cdots \wedge a_{m_1}) \otimes (a_1 \wedge a_2 \wedge \cdots \wedge a_{m_2}) \otimes \cdots \otimes (a_1 \wedge a_2 \wedge \cdots \wedge a_{m_k}),$$

which is called the *highest weight vector of* $[n_1 n_2 \cdots n_l]$.

We define the following elements $X_{i,j}, Y_{i,j}$ $(i \neq j)$ and U_i, V_i of $\mathfrak{sp}(2g, \mathbb{Q})$ characterized by their actions on $H_{\mathbb{Q}}$ as follows:

$$\begin{array}{rclrcl} X_{i,j}(a_k) &=& \delta_{jk} a_i, & X_{i,j}(b_k) &=& -\delta_{ik} b_j, \\ Y_{i,j}(a_k) &=& 0, & Y_{i,j}(b_k) &=& \delta_{ik} a_j + \delta_{jk} a_i, \\ U_i(a_k) &=& 0, & U_i(b_k) &=& \delta_{ik} a_i, \\ V_i(a_k) &=& \delta_{ik} b_i, & V_i(b_k) &=& 0. \end{array}$$

We will frequently make use of the following $Sp(2g, \mathbb{Q})$-(and $\mathfrak{sp}(2g, \mathbb{Q})$-) equivariant homomorphisms.

1) The *contraction* $C_k^{(i,j)} : \otimes^k H_{\mathbb{Q}} \to \otimes^{k-2} H_{\mathbb{Q}}$ $(1 \leq i < j \leq k)$ is given by

$$C_k^{(i,j)}(x_1 \otimes \cdots \otimes x_k) = \mu(x_i, x_j) x_1 \otimes \cdots \otimes \widehat{x_i} \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_k,$$

where $\widehat{x_i}$ means excluding x_i, and $\otimes^0 H_{\mathbb{Q}} = \mathbb{Q}$ is the trivial representation.

2) For each symbol
$$\sigma = (i(1,1),\dots,i(1,k_1))(i(2,1),\dots,i(2,k_2))\cdots(i(l,1),\dots,i(l,k_l)),$$
where $k = \sum_{j=1}^{l} k_j$ and $\{i(1,1),\dots,i(1,k_1),i(2,1),\dots,i(l,k_l)\} = \{1,2,\dots,k\}$, the projection $p_k^\sigma : \otimes^k H_{\mathbb{Q}} \to (\wedge^{k_1} H_{\mathbb{Q}}) \otimes (\wedge^{k_2} H_{\mathbb{Q}}) \otimes \cdots \otimes (\wedge^{k_l} H_{\mathbb{Q}})$ is given by
$$p_k^\sigma(x_1 \otimes \cdots \otimes x_k) = (x_{i(1,1)} \wedge \cdots \wedge x_{i(1,k_1)}) \otimes \cdots \otimes (x_{i(l,1)} \wedge \cdots \wedge x_{i(l,k_l)}).$$
For example, $p_3^{(1,3)(2)}(x_1 \otimes x_2 \otimes x_3) = (x_1 \wedge x_3) \otimes x_2$.

3) The canonical inclusion $\iota_k : \mathcal{L}_{g,1}^{\mathbb{Q}}(k) := \mathcal{L}_{g,1}(k) \otimes \mathbb{Q} \hookrightarrow H_{\mathbb{Q}}^{\otimes k}$, where $\mathcal{L}_{g,1}(k)$ is the degree k part of the free Lie algebra $\mathcal{L}_{g,1}$ generated by H, is inductively defined by replacing the bracket $[X,Y]$ by $X \otimes Y - Y \otimes X$.

2.4. **Johnson's homomorphisms via the representation theory of $Sp(2g,\mathbb{Q})$.** Let $\tau_g^{\mathbb{Q}}(k)$ denote the Johnson homomorphism $\tau_g(k)$ tensored by $\mathbb{Q}$, namely
$$\tau_g^{\mathbb{Q}}(k) : (\mathcal{M}_g[k+1]/(\Gamma^2\mathcal{M}_g[k+1])) \otimes \mathbb{Q} \longrightarrow \mathfrak{h}_g^{\mathbb{Q}}(k) := \mathfrak{h}_g(k) \otimes \mathbb{Q}.$$
As mentioned in Section 2.2, $\tau_g^{\mathbb{Q}}(k)$ is $\mathcal{M}_g$-equivariant, and in particular, $\operatorname{Im}\tau_g^{\mathbb{Q}}(k)$ is an $Sp(2g,\mathbb{Z})$-vector space. Moreover, it turns out that $\operatorname{Im}\tau_g^{\mathbb{Q}}(k)$ is in fact an $Sp(2g,\mathbb{Q})$-vector space by Asada-Nakamura [3, Lemma 2.2.8]. Similar results hold for $\tau_{g,*}^{\mathbb{Q}}(k) := \tau_{g,*}(k) \otimes \mathbb{Q}$ and $\tau_{g,1}^{\mathbb{Q}}(k) := \tau_{g,1}(k) \otimes \mathbb{Q}$. By results of Johnson [10] for $k=1$, Hain [9] and Morita [18] for $k=2$, Hain [9] and Asada-Nakamura [3] for $k=3$, we have the following.
$$\begin{aligned}
&\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(1) = \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(1) = \mathfrak{h}_{g,1}^{\mathbb{Q}}(1) = \mathfrak{h}_{g,*}^{\mathbb{Q}}(1) = [1^3]+[1] = \wedge^3 H_{\mathbb{Q}},\\
&\operatorname{Im}\tau_{g}^{\mathbb{Q}}(1) \ = \mathfrak{h}_{g}^{\mathbb{Q}}(1) \ = [1^3] = (\wedge^3 H_{\mathbb{Q}})/H_{\mathbb{Q}},\\
&\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(2) = \mathfrak{h}_{g,1}^{\mathbb{Q}}(2) = [2^2]+[1^2]+[0],\\
&\operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(2) = \mathfrak{h}_{g,*}^{\mathbb{Q}}(2) = [2^2]+[1^2],\\
&\operatorname{Im}\tau_{g}^{\mathbb{Q}}(2) \ = \mathfrak{h}_{g}^{\mathbb{Q}}(2) \ = [2^2]\\
&\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(3) = \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(3) = [31^2]+[21] \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(3) = \mathfrak{h}_{g,*}^{\mathbb{Q}}(3) = [31^2]+[21]+[3],\\
&\operatorname{Im}\tau_{g}^{\mathbb{Q}}(3) \ = [31^2] \subset \mathfrak{h}_{g}^{\mathbb{Q}}(3) = [31^2]+[3]
\end{aligned}$$
for $g \geq 3$. Moreover, in [21], Morita announced that
$$\begin{aligned}
&\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4) = \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(4) = [42]+[31^3]+2[31]+[2^3]+[21^2]+2[2],\\
&\operatorname{Im}\tau_{g}^{\mathbb{Q}}(4) \ = [42]+[31^3]+[31]+[2^3]+[2]
\end{aligned}$$
in
$$\begin{aligned}
\mathfrak{h}_{g,1}^{\mathbb{Q}}(4) &= [42]+[31^3]+2[31]+[2^3]+2[21^2]+3[2],\\
\mathfrak{h}_{g,*}^{\mathbb{Q}}(4) &= [42]+[31^3]+2[31]+[2^3]+2[21^2]+2[2],\\
\mathfrak{h}_{g}^{\mathbb{Q}}(4) &= [42]+[31^3]+[31]+[2^3]+[21^2]+[2]
\end{aligned}$$
for $g \geq 4$.

Remark 2.1. Hain showed in [9] that as Lie algebras, $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}, \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}, \operatorname{Im}\tau_{g}^{\mathbb{Q}}$ are generated by their degree 1 parts, and that $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(k) \cong \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(k)$ for $k \geq 3$.

Remark 2.2. A part of our proof of Theorems 4.2, 5.1 in later sections has overlaps with Morita's proof [22] of the determinations of $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4) \cong \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(4)$ and $\operatorname{Im}\tau_{g}^{\mathbb{Q}}(4)$. More precisely, we will prove that the summands displayed above are contained in $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4)$, $\operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(4)$ and $\operatorname{Im}\tau_{g}^{\mathbb{Q}}(4)$ by using brackets of two elements in $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(2)$, $\operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(2)$ and $\operatorname{Im}\tau_{g}^{\mathbb{Q}}(2)$. As a result, we can see that the degree 4 parts of the rational images of Johnson's homomorphisms are generated by their degree 2 parts.

The bracket operation of $\mathfrak{h}_{g,1}$ is explicitly given in [19, 20]. In this paper, however, we use an alternative description given by Garoufalidis-Levine [8], Levine [14, 15], which will be easier to handle. We recall the following Lie algebra of labeled unitrivalent trees. Let $\mathcal{A}_k^t(H)$ be the abelian group generated by unitrivalent trees with $k+2$ univalent vertices labeled by elements of H and a cyclic order of each trivalent vertex modulo relations of AS and IHX together with linearity of labels. We can endow $\mathcal{A}^t(H) := \{\mathcal{A}_k^t(H)\}_{k\geq 1}$ with a bracket operation

$$[\,\cdot\,,\,\cdot\,] : \mathcal{A}_k^t(H)\otimes\mathcal{A}_l^t(H) \longrightarrow \mathcal{A}_{k+l}^t(H)$$

given below, and $\mathcal{A}^t(H)$ becomes a quasi Lie algebra. For labeled trees $T_1, T_2 \in \mathcal{A}^t(H)$, we define

$$[T_1, T_2] := \sum_{i,j} \mu(a_i, b_j) T_1 *_{i,j} T_2$$

where the sum is taken over all pairs of a univalent vertex of T_1, labeled by a_i, and one of T_2, labeled by b_j, and $T_1 *_{i,j} T_2$ is the tree given by welding T_1 and T_2 at the pair. We define a map $\eta_k : \mathcal{A}_k^t(H) \to H\otimes\mathcal{L}_{g,1}(k+1)$ by

$$\eta_k(T) := \sum_v a_v \otimes T_v,$$

where the sum is over all univalent vertices of T, and for each univalent vertex v, a_v denotes the label of v and T_v denotes the rooted labeled planar binary tree obtained by removing the label a_v and considering v to be an unlabeled root, which can be regarded as an element of $\mathcal{L}_{g,1}(k+1)$ by a standard method. It is shown that $\eta := \{\eta_k\}_{k\geq 1} : \mathcal{A}^t(H) \to H\otimes\mathcal{L}_{g,1}$ is a quasi Lie algebra homomorphism and $\operatorname{Im}\eta \subset \mathfrak{h}_{g,1}$. Moreover $\eta\otimes\mathbb{Q} : \mathcal{A}^t(H)\otimes\mathbb{Q} \to \mathfrak{h}_{g,1}^{\mathbb{Q}}$ becomes an isomorphism of Lie algebras. For more details, see [8], [14, 15] and their references. In what follows, we identify $\mathcal{A}^t(H)\otimes\mathbb{Q}$ with $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ by $\eta\otimes\mathbb{Q}$.

Using $\mathcal{A}^t(H)\otimes\mathbb{Q}$, we now give a graphical description of the map

$$\Psi_k : \mathcal{L}_g^{\mathbb{Q}}(k) := \mathcal{L}_g(k)\otimes\mathbb{Q} \hookrightarrow \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(k) \subset \mathfrak{h}_{g,*}^{\mathbb{Q}}(k),$$

which was mentioned in Section 2.2 and is explicitly given by

$$\mathcal{L}_g^{\mathbb{Q}}(k) \ni X \;\mapsto\; \sum_{i=1}^{g} \big(a_i\otimes[b_i, X] - b_i\otimes[a_i, X]\big) \in H\otimes\mathcal{L}_g^{\mathbb{Q}}(k+1),$$

as follows. For each rooted labeled planar binary tree T as an element of $\mathcal{L}_{g,1}^{\mathbb{Q}}(k)$, we can construct an element of $\mathcal{A}_k^t(H)\otimes\mathbb{Q} \cong \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ by gluing T to the rooted labeled planar binary

tree $T_{\omega_0} \in \mathcal{L}_{g,1}^{\mathbb{Q}}(2) \cong \wedge^2 H_{\mathbb{Q}}$ corresponding to $\omega_0 = \sum_{i=1}^{g} a_i \wedge b_i$ at their roots as depicted in Figure 1.

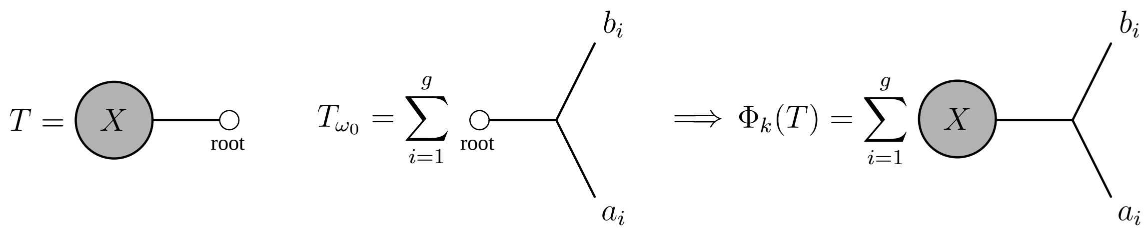


Figure 1. The map $\Phi_k : \mathcal{L}_{g,1}^{\mathbb{Q}}(k) \to \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$

It can be easily checked that this construction gives an $Sp(2g,\mathbb{Q})$-equivariant homomorphism $\Phi_k : \mathcal{L}_{g,1}^{\mathbb{Q}}(k) \longrightarrow \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$, and it induces the desired map $\Psi_k : \mathcal{L}_{g}^{\mathbb{Q}}(k) \hookrightarrow \mathfrak{h}_{g,*}^{\mathbb{Q}}(k)$ by using Labute's result [13].

3. Description of the second Johnson homomorphism

In this paper, we mainly concern the second Johnson homomorphism and the map induced on the rational cohomology. Now we see the second Johnson homomorphism more closely. Recall that

$$\begin{tikzcd}
\mathcal{K}_{g,1} \arrow[r, "\tau_{g,1}^{\mathbb{Q}}(2)"] \arrow[d] & \mathfrak{h}_{g,1}^{\mathbb{Q}}(2) \arrow[r, equal] \arrow[d] & {[2^2]+[1^2]+[0]} \arrow[d, "\mathrm{proj}"] \\
\mathcal{K}_{g,*} \arrow[r, "\tau_{g,*}^{\mathbb{Q}}(2)"] \arrow[d] & \mathfrak{h}_{g,*}^{\mathbb{Q}}(2) \arrow[r, equal] \arrow[d] & {[2^2]+[1^2]} \arrow[d, "\mathrm{proj}"] \\
\mathcal{K}_{g} \arrow[r, "\tau_{g}^{\mathbb{Q}}(2)"] & \mathfrak{h}_{g}^{\mathbb{Q}}(2) \arrow[r, equal] & {[2^2]}
\end{tikzcd}$$

for $g \geq 2$. Note that by passing to the duals, we have $Sp(2g,\mathbb{Q})$-equivariant inclusions

$$[2^2] \hookrightarrow [2^2]+[1^2] \hookrightarrow [2^2]+[1^2]+[0]$$

which are unique up to scalars.

The summands $[0]$ and $[1^2]$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(2)$ can be detected by the composite of maps

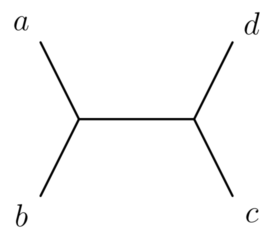
$$q_{[1^2]} : \mathfrak{h}_{g,1}^{\mathbb{Q}}(2) \hookrightarrow H \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(3) \overset{1\otimes\iota_3}{\hookrightarrow} H_{\mathbb{Q}}^{\otimes 4} \xrightarrow{C_4^{(1,2)}} H_{\mathbb{Q}}^{\otimes 2} \xrightarrow{p_2^{(1,2)}} \wedge^2 H_{\mathbb{Q}},$$
$$q_{[0]} : \mathfrak{h}_{g,1}^{\mathbb{Q}}(2) \hookrightarrow H \otimes \mathcal{L}_{g,1}^{\mathbb{Q}}(3) \overset{1\otimes\iota_3}{\hookrightarrow} H_{\mathbb{Q}}^{\otimes 4} \xrightarrow{C_2^{(1,2)}\circ C_4^{(1,2)}} \mathbb{Q} = [0],$$

respectively. On the other hand, by using the map $\Phi_2 : \mathcal{L}_{g,1}^{\mathbb{Q}}(2) = \wedge^2 H_{\mathbb{Q}} = [1^2]+[0] \to \mathfrak{h}_{g,1}^{\mathbb{Q}}(2)$ mentioned in Section 2.4, we can obtain the highest weight vectors $v_{[2^2]}$, $v_{[1^2]}$ and $v_{[0]}$ of $[2^2]$, $[1^2]$ and $[0]$ defined by

$$v_{[2^2]} = T^H(a_1,a_2,a_1,a_2), \quad v_{[1^2]} = \sum_{i=1}^{g} T^H(a_1,a_2,a_i,b_i), \quad v_{[0]} = \sum_{i,j=1}^{g} T^H(a_i,b_i,a_j,b_j),$$

where we put

$$T^H(a,b,c,d) :=$$



$$\in \mathfrak{h}_{g,1}^{\mathbb{Q}}(2).$$

In [18], Morita computed the value of the Dehn twist $\psi_h \in \mathcal{K}_{g,1}$ along the simple closed curve γ_h bounding the surface $\Sigma_{h,1}$ of the standard position by $\tau_{g,1}^{\mathbb{Q}}(2)$, and he obtained that

$$\begin{aligned}\tau_{g,1}^{\mathbb{Q}}(2)(\psi_h) &= \sum_{i,j=1}^{h} \left\{a_i \otimes [[a_j,b_j],b_i] - b_i \otimes [[a_j,b_j],a_i]\right\} \\ &= -\frac{1}{2} \sum_{i,j=1}^{h} T^H(a_i,b_i,a_j,b_j).\end{aligned}$$

In particular, we have $\tau_{g,1}^{\mathbb{Q}}(2)(\psi_1^{-2}) = T^H(a_1,b_1,a_1,b_1)$.

Let $\overline{\psi_1^{-2}} \in \mathcal{K}_g$ be the image of $\psi_1^{-2} \in \mathcal{K}_{g,1}$ under the projection $\mathcal{K}_{g,1} \to \mathcal{K}_g$. To compute $\tau_g^{\mathbb{Q}}(2)\big(\overline{\psi_1^{-2}}\big)$, we need to project the vector $\tau_{g,1}^{\mathbb{Q}}(2)(\psi_1^{-2}) = T^H(a_1,b_1,a_1,b_1)$ onto $[2^2] \subset [2^2] + [1^2] + [0]$. By direct computations, we can see that

$$\begin{aligned}
&q_{[1^2]}(T^H(a_1,b_1,a_1,b_1)) = 12 a_1 \wedge b_1, && q_{[0]}(T^H(a_1,b_1,a_1,b_1)) = 12,\\
&q_{[1^2]}(\Phi_2(a_1 \wedge b_1)) = (4g+4) a_1 \wedge b_1 + 4\omega_0, && q_{[0]}(\Phi_2(a_1 \wedge b_1)) = 8g+4,\\
&q_{[1^2]}(\Phi_2(\omega_0)) = q_{[1^2]}(v_{[0]}) = (8g+4)\omega_0, && q_{[0]}(\Phi_2(\omega_0)) = q_{[0]}(v_{[0]}) = 8g^2+4g,
\end{aligned}$$

and therefore we have

$$\begin{aligned}\tau_g^{\mathbb{Q}}(2)\big(\overline{\psi_1^{-2}}\big) &= T^H(a_1,b_1,a_1,b_1) - \frac{3}{g+1}\Phi_2(a_1 \wedge b_1) + \frac{3}{(2g+1)(g+1)}\Phi_2(\omega_0) \\ &\in [2^2] = \mathfrak{h}_g(2).\end{aligned}$$

Consequently, we have the following.

Proposition 3.1. (1) (Morita [18]) $\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(2)$ *contains the vector* $T^H(a_1,b_1,a_1,b_1)$.
(2) $\operatorname{Im} \tau_g^{\mathbb{Q}}(2)$ *contains the vector*

$$T^H(a_1,b_1,a_1,b_1) - \frac{3}{g+1}\sum_{i=1}^{g} T^H(a_1,b_1,a_i,b_i) + \frac{3}{(2g+1)(g+1)}\sum_{i,j=1}^{g} T^H(a_i,b_i,a_j,b_j).$$

4. Computation for the case of a surface with a boundary

4.1. **Statement for $\mathcal{K}_{g,1}$.** We begin our proof of the main theorem. First, we treat the case of $\mathcal{K}_{g,1}$, whose computational results will be used again for proofs of the cases of $\mathcal{K}_{g,*}$ and $\mathcal{K}_g$.

As seen in Section 2.4, we have $\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(2) = \mathfrak{h}_{g,1}^{\mathbb{Q}}(2) \cong [2^2] + [1^2] + [0]$, so that an injection $[2^2] + [1^2] + [0] \hookrightarrow H^1(\mathcal{K}_{g,1};\mathbb{Q})$ is obtained. We now consider the cup product map

$$\cup : \wedge^2([2^2] + [1^2] + [0]) \longrightarrow H^2(\mathcal{K}_{g,1};\mathbb{Q}).$$

Note that

$$\wedge^2([2^2]+[1^2]+[0]) \cong \wedge^2[2^2]+[2^2]\otimes[1^2]+\wedge^2[1^2]+[2^2]\otimes[0]+[1^2]\otimes[0].$$

Lemma 4.1. *For $g \geq 4$, the irreducible decompositions of $\wedge^2[2^2]$, $[2^2]\otimes[1^2]$, $\wedge^2[1^2]$, $[2^2]\otimes[0]$ and $[1^2]\otimes[0]$ are given by the following table.*

	$\wedge^2[2^2]$	$[2^2]\otimes[1^2]$	$\wedge^2[1^2]$	$[2^2]\otimes[0]$	$[1^2]\otimes[0]$
$[431]$	1				
$[42]$	1				
$[32^21]$	1				
$[321]$	1	1			
$[31^3]$	1				
$[31]$	1	1			
$[2^3]$	1				
$[21^2]$	1	1	1		
$[2]$	1		1		
$[3^2]$		1			
$[2^21^2]$		1			
$[2^2]$		1		1	
$[1^2]$		1			1

Table 1. *Irreducible decompositions of $Sp(2g,\mathbb{Q})$-modules for $g \geq 4$*

In particular, we have

$$\begin{aligned}\wedge^2([2^2]+[1^2]+[0]) \cong &([431]+[42]+[32^21]+[321]+[31^3]+[31]+[2^3]+[21^2]+[2])\\ &+([321]+[31]+[21^2]+[3^2]+[2^21^2]+[2^2]+[1^2])\\ &+([21^2]+[2])+[2^2]+[1^2].\end{aligned}$$

Proof. By a general theory of the representation (see [9, Remark 6.2]), it is observed that the irreducible decompositions we now consider become stable for $g \geq 4$. The computations in the next subsections show that the summands in the table are actually contained. To show that they are all, it suffices to check that the total dimension of the summands coincides with the dimension of $\wedge^2\mathfrak{h}_{g,1}^{\mathbb{Q}}(2)$ in the case $g=4$ by using Weyl's character formula (see [7, Section 24.2]). We now omit the details. $\square$

The first main theorem of this section is the following.

Theorem 4.2. *For $g \geq 4$, the kernel of the cup product map $\cup : \wedge^2([2^2]+[1^2]+[0]) \to H^2(\mathcal{K}_{g,1};\mathbb{Q})$ is*

$$[42]+[31^3]+2[31]+[2^3]+[21^2]+2[2],$$

which is, as an $Sp(2g,\mathbb{Q})$-vector space, isomorphic to $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4)$.

This theorem follows from Lemmas 4.4, 4.6 in the next two subsections .

4.2. **A lower bound of the kernel.** Here we show that the summands in Theorem 4.2 are contained in the kernel of the cup product map. Note that the map $\cup : \wedge^2([2^2]+[1^2]+[0]) \to H^2(\mathcal{K}_{g,1};\mathbb{Q})$ is nothing other than the homomorphism

$$\tau_{g,1}(2)^* : H^2(\operatorname{Im}\tau_{g,1}(2);\mathbb{Q}) \longrightarrow H^2(\mathcal{K}_{g,1};\mathbb{Q}).$$

Hence, by passing to the dual, our task is equivalent to observing the cokernel of the map

$$\tau_{g,1}(2)_* : H_2(\mathcal{K}_{g,1};\mathbb{Q}) \longrightarrow H_2(\operatorname{Im}\tau_{g,1}(2);\mathbb{Q}).$$

By applying Stallings' exact sequence [25] to the group extension

$$1 \longrightarrow \Gamma^2(\mathcal{K}_{g,1}) \longrightarrow \mathcal{K}_{g,1} \longrightarrow H_1(\mathcal{K}_{g,1}) \longrightarrow 1$$

and observing the homomorphisms, we obtain the exact sequence

$$H_2(\mathcal{K}_{g,1};\mathbb{Q}) \longrightarrow \wedge^2 H_1(\mathcal{K}_{g,1};\mathbb{Q}) \xrightarrow{[\cdot,\cdot]} ((\Gamma^2\mathcal{K}_{g,1})/(\Gamma^3\mathcal{K}_{g,1}))\otimes\mathbb{Q} \longrightarrow 0$$

where the first map is the coproduct on the rational homology and the second one is the Lie bracket

$$[\,\cdot\,,\,\cdot\,] : \wedge^2 H_1(\mathcal{K}_{g,1}) = \wedge^2((\Gamma^1\mathcal{K}_{g,1})/(\Gamma^2\mathcal{K}_{g,1})) \longrightarrow (\Gamma^2\mathcal{K}_{g,1})/(\Gamma^3\mathcal{K}_{g,1}).$$

Since $\Gamma^2\mathcal{K}_{g,1} \subset \mathcal{M}_{g,1}[5]$ and $\Gamma^3\mathcal{K}_{g,1} \subset \mathcal{M}_{g,1}[7]$, we have a natural map

$$i : (\Gamma^2\mathcal{K}_{g,1})/(\Gamma^3\mathcal{K}_{g,1}) \longrightarrow \mathcal{M}_{g,1}[5]/\mathcal{M}_{g,1}[6].$$

By considering the fact that $\tau_{g,1} : \{\mathcal{M}_{g,1}[k+1]/\mathcal{M}_{g,1}[k+2]\}_{k\geq 1} \to \operatorname{Im}\tau_{g,1} \subset \mathfrak{h}_{g,1}$ is a Lie algebra homomorphism as mentioned in Section 2.2, we have the commutative diagram

$$\begin{tikzcd}
\wedge^2 H_1(\mathcal{K}_{g,1}) \arrow[r, "{[\cdot,\cdot]}"] \arrow[d, "\wedge^2\tau_{g,1}(2)"'] & (\Gamma^2\mathcal{K}_{g,1})/(\Gamma^3\mathcal{K}_{g,1}) \arrow[r, "i"] & \mathcal{M}_{g,1}[5]/\mathcal{M}_{g,1}[6] \arrow[d, "\tau_{g,1}(4)", "\cong"'] \\
\wedge^2 \operatorname{Im}\tau_{g,1}(2) \arrow[r, "{[\cdot,\cdot]}"] & \operatorname{Im}\tau_{g,1}(4) \arrow[r, equal] & \operatorname{Im}\tau_{g,1}(4).
\end{tikzcd}$$

Consequently, we obtain the commutative diagram

$$\begin{tikzcd}
H_2(\mathcal{K}_{g,1};\mathbb{Q}) \arrow[r] \arrow[d, "\tau_{g,1}(2)_*"'] & \wedge^2 H_1(\mathcal{K}_{g,1};\mathbb{Q}) \arrow[r, "{[\cdot,\cdot]}"] \arrow[d, "\wedge^2\tau^{\mathbb{Q}}_{g,1}(2)"'] & ((\Gamma^2\mathcal{K}_{g,1})/(\Gamma^3\mathcal{K}_{g,1}))\otimes\mathbb{Q} \arrow[d, "\tau^{\mathbb{Q}}_{g,1}(4)\circ i"] \\
H_2(\operatorname{Im}\tau_{g,1}(2);\mathbb{Q}) \arrow[r, "\cong"] & \wedge^2 \operatorname{Im}\tau^{\mathbb{Q}}_{g,1}(2) \arrow[r, "{[\cdot,\cdot]}"] & \operatorname{Im}\tau^{\mathbb{Q}}_{g,1}(4)
\end{tikzcd}$$

whose upper row is exact. Hence we can find some summands in the cokernel of $\tau_{g,1}(2)_* : H_2(\mathcal{K}_{g,1};\mathbb{Q}) \to H_2(\operatorname{Im}\tau_{g,1}(2);\mathbb{Q})$ by observing the image of

$$[\,\cdot\,,\,\cdot\,] \;:\; \wedge^2 \operatorname{Im}\tau^{\mathbb{Q}}_{g,1}(2) \longrightarrow \operatorname{Im}\tau^{\mathbb{Q}}_{g,1}(4).$$

Remark 4.3. It is not known whether $H_1(\mathcal{K}_{g,1};\mathbb{Q})$ is finite dimensional or not. If it is finite dimensional, we could use Sullivan's exact sequence [26] in the above argument as in Pettet's one [23, Section 3.2].

Lemma 4.4. *For $g \geq 4$, the image of the Lie bracket $[\cdot,\cdot] : \wedge^2 \operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(2) \to \operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(4)$ contains*

$$[42] + [31^3] + 2[31] + [2^3] + [21^2] + 2[2].$$

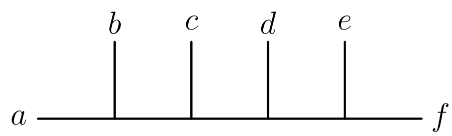
Hence, these summands are contained in the kernel of the cup product map.

Proof. We prove this by direct computations.

$[42]$: Consider $v_{[2^2]} = T^H(a_1, a_2, a_1, a_2) \in [2^2]$ and $\frac{1}{2}V_2(v_{[2^2]}) = T^H(a_1, b_2, a_1, a_2) \in [2^2]$. Then

$$[T^H(a_1, a_2, a_1, a_2), T^H(a_1, b_2, a_1, a_2)] = 2T(a_1, a_2, a_1, a_1, a_2, a_1),$$

where we put

$T(a, b, c, d, e, f) :=$ 

$\in \mathfrak{h}_{g,1}^{\mathbb{Q}}(4).$

Then we have

$$p_6^{(1,2)(3,4)(5)(6)} \circ \iota_6(2T(a_1, a_2, a_1, a_1, a_2, a_1)) = -60(a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes a_1 \otimes a_1.$$

This is the highest weight vector of $[42]$, so that $[42]$ is contained in $\operatorname{Im}[\cdot,\cdot]$ for $g \geq 2$.

$[31^3]$: Consider $T^H(a_1, b_2, a_1, a_2),\ T^H(a_1, a_2, a_3, a_4) \in [2^2]$. Then

$$[T^H(a_1, b_2, a_1, a_2), T^H(a_1, a_2, a_3, a_4)] = T(a_1, a_2, a_1, a_1, a_4, a_3)$$

Applying $p_6^{(1,2,3,4)(5)(6)} \circ \iota_6$, we have

$$-12(a_1 \wedge a_2 \wedge a_3 \wedge a_4) \otimes a_1 \otimes a_1.$$

This is the highest weight vector of $[31^3]$, so that $[31^3]$ is contained in $\operatorname{Im}[\cdot,\cdot]$ for $g \geq 4$.

$[2^3]$: Consider $T^H(a_3, a_2, a_3, a_2) \in [2^2]$. Then

$$[T^H(a_3, a_2, a_3, a_2), T^H(a_1, b_2, a_1, a_2)] = 2T(a_3, a_2, a_3, a_1, a_2, a_1)$$

Applying $p_6^{(1,2,3)(4,5,6)} \circ \iota_6$, we have

$$-72(a_1 \wedge a_2 \wedge a_3) \otimes (a_1 \wedge a_2 \wedge a_3).$$

This is the highest weight vector of $[2^3]$, so that $[2^3]$ is contained in $\operatorname{Im}[\cdot,\cdot]$ for $g \geq 3$.

$[31]$: Consider the vectors $T^H(a_1, a_2, b_3, a_2),\ T^H(a_1, b_2, a_1, a_3),\ T^H(a_3, a_2, a_3, a_2) \in [2^2]$ and $T^H(a_1, b_1, a_1, b_2) \in [2^2] + [1^2]$. Then we have

$$\begin{aligned}
\xi_1 :&= [T^H(a_1, a_2, b_3, a_2), T^H(a_1, b_2, a_1, a_3)] \\
&= T(a_1, a_2, b_3, a_1, a_3, a_1) + T(b_3, a_2, a_1, a_1, a_3, a_1) + T(a_1, a_2, a_2, a_1, b_2, a_1), \\
\xi_2 :&= \frac{1}{2}[T^H(a_1, a_2, a_1, a_2), T^H(a_1, b_1, a_1, b_2)] \\
&= T(a_2, a_1, a_2, a_1, b_2, a_1) + T(a_1, b_1, a_1, a_1, a_2, a_1)
\end{aligned}$$

and

	ξ_1	ξ_2
$p_4^{(1,2)(3)(4)} \circ C_6^{(1,2)} \circ \iota_6$	0	$-12(a_1 \wedge a_2) \otimes a_1 \otimes a_1$
$p_4^{(1,2)(3)(4)} \circ C_6^{(1,3)} \circ \iota_6$	$4(a_1 \wedge a_2) \otimes a_1 \otimes a_1$	$-4(a_1 \wedge a_2) \otimes a_1 \otimes a_1$

Note that ξ_1 are defined for $g \geq 3$ and ξ_2 are for $g \geq 2$. From the table, we see that $\mathrm{Im}[\cdot, \cdot]$ contains $[31]$ for $g = 2$, and $2[31]$ for $g \geq 3$.

$[2]$: Consider the vectors $\frac{1}{4}V_1V_2^2(v_{[2^2]}) = T^H(a_1, b_2, b_1, b_2) \in [2^2]$ and $T^H(a_1, b_1, a_1, b_2)$, $T^H(a_1, a_2, a_1, b_1) \in [2^2] + [1^2]$. Then we have

$$\begin{aligned}
\xi_1 &:= \frac{1}{2}[T^H(a_1, a_2, a_1, a_2), T^H(a_1, b_2, b_1, b_2)] \\
&= T(a_1, b_2, b_2, a_2, a_2, a_1) + T(a_1, a_2, a_1, a_1, b_2, b_1) + T(a_1, b_2, b_1, a_1, a_2, a_1), \\
\xi_2 &:= [T^H(a_1, b_1, a_1, b_2), T^H(a_1, a_2, a_1, b_1)] \\
&= T(b_2, a_1, b_1, a_1, a_2, a_1) + T(b_1, a_1, b_2, a_1, a_2, a_1) + T(a_1, a_2, b_1, a_1, b_2, a_1) \\
&\quad + T(a_1, b_1, a_2, a_1, b_2, a_1) + T(b_1, a_1, a_1, a_1, b_1, a_1)
\end{aligned}$$

and

	ξ_1	ξ_2
$C_4^{(1,2)} \circ C_6^{(1,2)} \circ \iota_6$	0	$18a_1 \otimes a_1$
$C_4^{(1,3)} \circ C_6^{(1,2)} \circ \iota_6$	$6a_1 \otimes a_1$	$-6a_1 \otimes a_1$
$C_4^{(1,2)} \circ C_6^{(1,3)} \circ \iota_6$	$-6a_1 \otimes a_1$	$6a_1 \otimes a_1$

Note that ξ_1, ξ_2 are defined for $g \geq 2$. From the table, we see that $\mathrm{Im}[\cdot, \cdot]$ contains $2[2]$ for $g \geq 2$.

$[21^2]$: Consider $\Phi_2(a_3 \wedge b_2) = \sum_{i=1}^g T^H(a_3, b_2, a_i, b_i) \in [1^2]$. Then we have

$$\begin{aligned}
&\frac{1}{2}\left[T^H(a_1, a_2, a_1, a_2),\ \sum_{i=1}^g T^H(a_3, b_2, a_i, b_i)\right] \\
&= \sum_{i=1}^g T(a_1, a_2, a_1, a_3, b_i, a_i) + T(a_3, b_2, a_2, a_1, a_2, a_1) + T(a_2, a_1, a_2, a_1, b_2, a_3).
\end{aligned}$$

Applying $p_4^{(1,2,3)(4)} \circ C_6^{(1,2)} \circ \iota_6$, we have

$$6g(a_1 \wedge a_2 \wedge a_3) \otimes a_1.$$

This is the highest weight vector of $[21^2]$, so that $[21^2]$ is contained in $\mathrm{Im}[\cdot, \cdot]$ for $g \geq 3$. $\square$

4.3. **An upper bound of the kernel.** We can detect summands in $\mathrm{Im}(\cup : \wedge^2([2^2]+[1^2]+[0]) \to H^2(\mathcal{K}_{g,1}; \mathbb{Q}))$ by observing the dual

$$\tau_{g,1}(2)_* : H_2(\mathcal{K}_{g,1}; \mathbb{Q}) \longrightarrow \wedge^2([2^2] + [1^2] + [0]) = H_2(\mathfrak{h}_{g,1}(2); \mathbb{Q}).$$

For each pair (φ, ψ) of elements of $\mathcal{K}_{g,1}$ which are commutative, we have a homomorphism $f : \mathbb{Z}^2 \to \mathcal{K}_{g,1}$ sending the standard generators of $\mathbb{Z}^2$ to φ and ψ. Then we can construct an element of $H_2(\mathcal{K}_{g,1}; \mathbb{Q})$ by considering the image of the generator of $1 \in \mathbb{Q} \cong H_2(\mathbb{Z}^2; \mathbb{Q})$ by f_*. Such a class is called an *abelian cycle*. The following is easily proved (see [24, Section 2]).

Lemma 4.5. *Let A be a finitely generated free abelian group and $f : \mathbb{Z}^2 \to A$ be a group homomorphism. Then the image of the generator $1 \in \mathbb{Q} \cong H_2(\mathbb{Z}^2;\mathbb{Q})$ by f_* is*

$$f(e_1) \wedge f(e_2) \in \wedge^2(A \otimes \mathbb{Q}) \cong H_2(A;\mathbb{Q}),$$

where e_1, e_2 are the standard generators of $\mathbb{Z}^2$

In our computation, we use this lemma for the image of each abelian cycle of $\mathcal{K}_{g,1}$ by $\tau_{g,1}(2)_*$.

Lemma 4.6. *For $g \geq 4$, $\mathrm{Im}(\tau_{g,1}(2)_* : H_2(\mathcal{K}_{g,1};\mathbb{Q}) \to H_2(\mathfrak{h}_{g,1}(2);\mathbb{Q}))$ contains*

$$[431] + [32^2 1] + 2[321] + 2[21^2] + [3^2] + [2^2 1^2] + 2[2^2] + 2[1^2].$$

Hence, these summands are not contained in the kernel of the cup product map.

Proof. We first show that the following elements

$$\begin{aligned} w_1 &:= T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_2,b_2,a_2,b_2) \\ w_2 &:= T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_3,b_3,a_3,b_3) \\ w_3 &:= T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_1,b_1,a_2,b_2) \end{aligned}$$

are contained in $\mathrm{Im}(\tau_{g,1}(2)_* : H_2(\mathcal{K}_{g,1};\mathbb{Q}) \to H_2(\mathfrak{h}_{g,1}(2);\mathbb{Q}))$. Indeed, the first two are easily obtained from Proposition 3.1. As for the last one, consider the abelian cycle corresponding to the pair of the elements $\psi_1, \psi_2 \in \mathcal{K}_{g,1}$ in Section 2.2. It is mapped by $\tau_{g,1}(2)_*$ to

$$\begin{aligned} &\frac{1}{4}T^H(a_1,b_1,a_1,b_1) \wedge \sum_{i,j=1}^{2} T^H(a_i,b_i,a_j,b_j) \\ &\qquad = \frac{1}{2}T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_1,b_1,a_2,b_2) + \frac{1}{4}T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_2,b_2,a_2,b_2), \end{aligned}$$

and our claim follows from this.

We decompose w_1, w_2, w_3 to each summands. We define a map

$$\iota : \wedge^2 \mathfrak{h}_{g,1}^{\mathbb{Q}}(2) \overset{\wedge^2\iota_4}{\hookrightarrow} \wedge^2(H_{\mathbb{Q}}^{\otimes 4}) \hookrightarrow H_{\mathbb{Q}}^{\otimes 8}$$

where the second inclusion sends $X \wedge Y$ to $X \otimes Y - Y \otimes X$ for $X, Y \in H_{\mathbb{Q}}^{\otimes 4}$.

$[431]$: Consider $w_1 = T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_2,b_2,a_2,b_2)$. Then we have

$$\begin{aligned} w_1 &\overset{X_{2,3}}{\longmapsto} -2T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_2,b_2,a_2,b_2) \\ &\overset{X_{1,2}^4}{\longmapsto} -48T^H(a_1,b_2,a_1,b_2) \wedge T^H(a_1,b_3,a_1,b_2) \\ &\overset{U_2^3}{\longmapsto} -288T^H(a_1,a_2,a_1,a_2) \wedge T^H(a_1,b_3,a_1,a_2) \\ &\overset{U_3}{\longmapsto} -288T^H(a_1,a_2,a_1,a_2) \wedge T^H(a_1,a_3,a_1,a_2). \end{aligned}$$

By $p_8^{(1,2,5)(4,3)(6,7)(8)} \circ \iota$, the last term is mapped to

$$20736(a_1 \wedge a_2 \wedge a_3) \otimes (a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes a_1,$$

which is the highest weight vector of $[431]$, so that $[431]$ is contained in $\mathrm{Im}\,\tau_{g,1}(2)_*$ for $g \geq 3$.

$[32^21]$: We have

$$\begin{aligned}&p_8^{(1,2,5,6)(3,4,7)(8)}\circ\iota(U_4X_{2,4}U_2U_3^2X_{1,2}X_{1,3}^2w_1)\\ =\;&p_8^{(1,2,5,6)(3,4,7)(8)}\circ\iota(-8T^H(a_1,a_3,a_1,a_3)\wedge T^H(a_1,a_2,a_2,a_4))\\ =\;&576(a_1\wedge a_2\wedge a_3\wedge a_4)\otimes(a_1\wedge a_2\wedge a_3)\otimes a_1,\end{aligned}$$

which is the highest weight vector of $[32^21]$, so that $[32^21]$ is contained in $\mathrm{Im}\,\tau_{g,1}(2)_*$ for $g\geq 4$.

$[321]$: Consider $w_1 = T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_2,b_2,a_2,b_2)$ and $w_3 = T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_1,b_1,a_2,b_2)$. Then we have

$$\begin{aligned}w_1 &\overset{U_2U_3^2X_{1,2}X_{1,3}^2}{\longmapsto} 8T^H(a_1,a_3,a_1,a_3)\wedge T^H(a_1,a_2,a_2,b_2)\\ &\longmapsto\quad 8T^H(a_1,a_2,a_1,a_2)\wedge T^H(a_1,a_3,a_3,b_3)\ =:\xi_1,\end{aligned}$$

where the second map is applying the element of $Sp(2g,\mathbb{Q})$ which acts on $H_{\mathbb{Q}}$ by

$$a_i\mapsto\begin{cases}a_3 & (i=2)\\ a_2 & (i=3)\\ a_i & (i\neq 2,3)\end{cases},\qquad b_i\mapsto\begin{cases}b_3 & (i=2)\\ b_2 & (i=3)\\ b_i & (i\neq 2,3)\end{cases},$$

and we denote it by $(a_2\leftrightarrow a_3, b_2\leftrightarrow b_3)\in Sp(2g,\mathbb{Q})$, for short. We also have

$$\begin{aligned}w_3 &\overset{U_2U_3^2X_{1,2}X_{1,3}^2}{\longmapsto} -4T^H(a_1,a_3,a_1,a_3)\wedge T^H(a_1,a_2,a_2,b_2)+4T^H(a_1,a_3,a_1,a_3)\wedge T^H(a_1,b_1,a_1,a_2)\\ &\qquad -8T^H(a_1,a_3,a_1,a_2)\wedge T^H(a_1,a_3,a_2,b_2)+8T^H(a_1,a_3,a_1,b_1)\wedge T^H(a_1,a_3,a_1,a_2)\\ &\overset{(a_2\leftrightarrow a_3,b_2\leftrightarrow b_3)}{\longmapsto} -4T^H(a_1,a_2,a_1,a_2)\wedge T^H(a_1,a_3,a_3,b_3)+4T^H(a_1,a_2,a_1,a_2)\wedge T^H(a_1,b_1,a_1,a_3)\\ &\qquad -8T^H(a_1,a_2,a_1,a_3)\wedge T^H(a_1,a_2,a_3,b_3)+8T^H(a_1,a_2,a_1,b_1)\wedge T^H(a_1,a_2,a_1,a_3)\\ &=:\xi_2,\end{aligned}$$

and

	ξ_1	ξ_2
$p_6^{(1,2,3)(4,5)(6)}\circ C_8^{(1,2)}\circ\iota$	$288(a_1\wedge a_2\wedge a_3)\otimes(a_1\wedge a_2)\otimes a_1$	$240(a_1\wedge a_2\wedge a_3)\otimes(a_1\wedge a_2)\otimes a_1$
$p_6^{(1,2,4)(3,5)(6)}\circ C_8^{(1,7)}\circ\iota$	0	$120(a_1\wedge a_2\wedge a_3)\otimes(a_1\wedge a_2)\otimes a_1$

Note that ξ_1,ξ_2 are defined for $g\geq 3$. From the table, we see that $\mathrm{Im}\,\tau_{g,1}(2)_*$ contains $2[321]$ for $g\geq 3$.

$[21^2]$: Consider w_1 and w_3. Then we have

$$\begin{aligned}w_1 &\overset{U_2U_3X_{1,2}X_{1,3}}{\longmapsto} 2T^H(a_1,a_2,a_1,a_3)\wedge T^H(a_2,b_2,a_2,b_2)-4T^H(a_1,b_1,a_1,a_3)\wedge T^H(a_1,a_2,a_2,b_2)\\ &=:\xi_1,\\ w_3 &\overset{U_2U_3X_{1,2}X_{1,3}}{\longmapsto} 2T^H(a_1,a_2,a_1,a_3)\wedge T^H(a_1,b_1,a_2,b_2)+2T^H(a_1,b_1,a_1,a_3)\wedge T^H(a_1,a_2,a_2,b_2)\\ &\qquad -2T^H(a_1,b_1,a_1,a_3)\wedge T^H(a_1,b_1,a_1,a_2)+2T^H(a_1,b_1,a_1,a_2)\wedge T^H(a_1,a_3,a_2,b_2)\\ &\qquad -T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_1,a_3,a_1,a_2)\\ &=:\xi_2,\end{aligned}$$

and

	ξ_1	ξ_2
$p_4^{(1,2,3)(4)} \circ C_6^{(5,6)} \circ C_8^{(1,2)} \circ \iota$	$-144(a_1 \wedge a_2 \wedge a_3) \otimes a_1$	$-48(a_1 \wedge a_2 \wedge a_3) \otimes a_1$
$p_4^{(1,2,3)(4)} \circ C_6^{(3,5)} \circ C_8^{(1,7)} \circ \iota$	0	$24(a_1 \wedge a_2 \wedge a_3) \otimes a_1$

Note that ξ_1, ξ_2 are defined for $g \geq 3$. From the table, we see that $\mathrm{Im}\,\tau_{g,1}(2)_*$ contains $2[21^2]$ for $g \geq 3$.

$[3^2]$: We have

$$w_1 \stackrel{U_2^3 X_{1,2}^3}{\longmapsto} -72T^H(a_1,a_2,a_1,b_1) \wedge T^H(a_1,a_2,a_1,a_2) + 72T^H(a_1,a_2,a_1,a_2) \wedge T^H(a_1,a_2,a_2,b_2)$$

Applying $p_6^{(1,2)(3,4)(5,6)} \circ C_8^{(1,2)} \circ \iota$, we obtain

$$-10368(a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes (a_1 \wedge a_2).$$

Hence $\mathrm{Im}\,\tau_{g,1}(2)_*$ contains $[3^2]$ for $g \geq 2$.

$[2^2 1^2]$: We have

$$\begin{aligned} w_1 &\stackrel{U_3X_{1,3}U_4X_{2,4}U_3X_{1,3}}{\longmapsto} -4T^H(a_1,a_3,a_1,a_3) \wedge T^H(a_2,b_2,a_2,a_4) \\ &\stackrel{(a_2 \leftrightarrow a_3, b_2 \leftrightarrow b_3)}{\longmapsto} -4T^H(a_1,a_2,a_1,a_2) \wedge T^H(a_3,b_3,a_3,a_4) \\ &\stackrel{p_6^{(1,2,3,4)(5,6)} \circ C_8^{(1,2)} \circ \iota}{\longmapsto} 288(a_1 \wedge a_2 \wedge a_3 \wedge a_4) \otimes (a_1 \wedge a_2). \end{aligned}$$

Hence $\mathrm{Im}\,\tau_{g,1}(2)_*$ contains $[2^2 1^2]$ for $g \geq 4$.

$[2^2]$: We have

$$\begin{aligned} w_2 &\stackrel{U_2^2 X_{1,2}^2}{\longmapsto} 4T^H(a_1,a_2,a_1,a_2) \wedge T^H(a_3,b_3,a_3,b_3) =: \xi_1, \\ w_3 &\stackrel{U_2^2 X_{1,2}^2}{\longmapsto} 4T^H(a_1,a_2,a_1,a_2) \wedge T^H(a_1,b_1,a_2,b_2) + 8T^H(a_1,b_1,a_1,a_2) \wedge T^H(a_1,a_2,a_2,b_2) \\ &\qquad - 4T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_1,a_2,a_1,a_2) \\ &\quad =: \xi_2, \end{aligned}$$

and

	ξ_1	ξ_2
$p_4^{(1,2)(3,4)} \circ C_6^{(1,2)} \circ C_8^{(1,2)} \circ \iota$	$-576(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$	$-960(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$
$p_4^{(1,2)(3,4)} \circ C_6^{(1,2)} \circ C_8^{(1,6)} \circ \iota$	0	$-240(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$

Note that ξ_1 are defined for $g \geq 3$, and ξ_2 for $g \geq 2$. From the table, we see that $\mathrm{Im}\,\tau_{g,1}(2)_*$ contains $[2^2]$ for $g = 2$, and $2[2^2]$ for $g \geq 3$.

$[1^2]$: We have

$$
\begin{aligned}
w_1 &\overset{U_2X_{1,2}}{\longmapsto} -2T^H(a_1,b_1,a_1,a_2)\wedge T^H(a_2,b_2,a_2,b_2)+2T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_1,a_2,a_2,b_2)\\
&=: \xi_1,\\
w_3 &\overset{U_2X_{1,2}}{\longmapsto} -2T^H(a_1,b_1,a_1,a_2)\wedge T^H(a_1,b_1,a_2,b_2)-T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_1,a_2,a_2,b_2)\\
&\qquad +T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_1,b_1,a_1,a_2)\\
&=: \xi_2,
\end{aligned}
$$

and

	ξ_1	ξ_2
$p_2^{(1,2)}\circ C_4^{(1,2)}\circ C_6^{(1,2)}\circ C_8^{(1,2)}\circ\iota$	$288a_1\wedge a_2$	$96a_1\wedge a_2$
$p_2^{(1,2)}\circ C_4^{(1,2)}\circ C_6^{(1,2)}\circ C_8^{(1,5)}\circ\iota$	0	$-48a_1\wedge a_2$

Note that ξ_1,ξ_2 are defined for $g\geq 3$. From the table, we see that $\operatorname{Im}\tau_{g,1}(2)_*$ contains $2[1^2]$ for $g\geq 2$. $\square$

From the above arguments, Theorem 4.2 follows.

4.4. **Statement for $\mathcal{K}_{g,*}$.** As seen in Section 2.4, we have $\operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(2)=\mathfrak{h}_{g,*}^{\mathbb{Q}}(2)\cong[2^2]+[1^2]$, so that an injection $[2^2]+[1^2]\hookrightarrow H^1(\mathcal{K}_{g,*};\mathbb{Q})$ is obtained. We now consider the cup product map

$$\cup:\wedge^2([2^2]+[1^2])\longrightarrow H^2(\mathcal{K}_{g,*};\mathbb{Q}).$$

Note that

$$
\begin{aligned}
\wedge^2([2^2]+[1^2]) &\cong \wedge^2\,[2^2]+[2^2]\otimes[1^2]+\wedge^2[1^2]\\
&\cong ([431]+[42]+[32^21]+[321]+[31^3]+[31]+[2^3]+[21^2]+[2])\\
&\quad +([321]+[31]+[21^2]+[3^2]+[2^21^2]+[2^2]+[1^2])+([21^2]+[2]).
\end{aligned}
$$

Theorem 4.7. *For $g\geq 4$, the kernel of the cup product map $\cup:\wedge^2([2^2]+[1^2])\to H^2(\mathcal{K}_{g,*};\mathbb{Q})$ is*

$$[42]+[31^3]+2[31]+[2^3]+[21^2]+2[2],$$

which is, as an $Sp(2g,\mathbb{Q})$-vector space, isomorphic to $\operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4)\cong\operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(4)$.

Proof. Since $\operatorname{Ker}(\wedge^2\mathfrak{h}_{g,1}^{\mathbb{Q}}(2)\to\wedge^2\mathfrak{h}_{g,*}^{\mathbb{Q}}(2))=1[2^2]+1[1^2]$, only we have to do is to observe the summands $[2^2]$ and $[1^2]$. In the last subsection, we have proved that $\operatorname{Im}\tau_{g,1}(2)_*$ contains $2[2^2]+2[1^2]$. Therefore $\operatorname{Im}\tau_{g,*}(2)_*$ certainly contains $1[2^2]+1[1^2]$, and this completes the proof. $\square$

5. Computation for the case of a closed surface

5.1. **Statement for $\mathcal{K}_g$.** As seen in Section 2.4, we have $\operatorname{Im}\tau_g^{\mathbb{Q}}(2)=\mathfrak{h}_g^{\mathbb{Q}}(2)\cong[2^2]$, so that an injection $[2^2]\hookrightarrow H^1(\mathcal{K}_g;\mathbb{Q})$ is obtained. We now consider the cup product map

$$\cup:\wedge^2[2^2]\longrightarrow H^2(\mathcal{K}_g;\mathbb{Q}).$$

Note that $\wedge^2[2^2]\cong[431]+[42]+[32^21]+[321]+[31^3]+[31]+[2^3]+[21^2]+[2]$.

Theorem 5.1. *For $g \geq 4$, the kernel of the cup product map $\cup : \wedge^2[2^2] \to H^2(\mathcal{K}_g;\mathbb{Q})$ is*

$$[42] + [31^3] + [31] + [2^3] + [2],$$

which is, as an $Sp(2g,\mathbb{Q})$-vector space, isomorphic to $\operatorname{Im}\tau_g^{\mathbb{Q}}(4)$.

This theorem follows from the arguments in the next two subsections.

5.2. **A lower bound of the kernel.** By an argument similar to that in Section 4.2, we can find some summands in $\wedge^2[2^2]$ which vanish in $H^2(\mathcal{K}_g;\mathbb{Q})$ by showing the following.

Lemma 5.2. *For $g \geq 4$, the image of the Lie bracket $[\cdot,\cdot] : \wedge^2 \operatorname{Im}\tau_g^{\mathbb{Q}}(2) \to \operatorname{Im}\tau_g^{\mathbb{Q}}(4)$ contains*

$$[42] + [31^3] + [31] + [2^3] + [2].$$

Proof. In the previous section, we have proved that

$$\operatorname{Im}([\cdot,\cdot] : \wedge^2 \operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(2) \to \operatorname{Im}\tau_{g,1}^{\mathbb{Q}}(4)) = [42] + [31^3] + 2[31] + [2^3] + [21^2] + 2[2] \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(4).$$

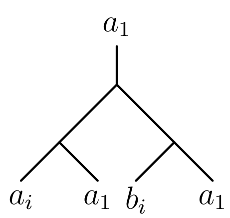
On the other hand, $\operatorname{Ker}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(4) \to \mathfrak{h}_g^{\mathbb{Q}}(4)) = [31] + [21^2] + 2[2]$. Hence at least $[42] + [31^3] + [31] + [2^3]$ is contained in $\operatorname{Im}[\cdot,\cdot] \subset \operatorname{Im}\tau_g^{\mathbb{Q}}(4)$. We now show that $[2]$ is also contained in $\operatorname{Im}[\cdot,\cdot]$.

The multiplicity of $[2]$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(4)$ is 3. We now prepare the following three vectors in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(4)$:

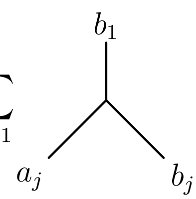
$$\begin{aligned}
\chi_1 &:= \frac{1}{2}[T^H(a_1,a_2,a_1,a_2), T^H(a_1,b_2,b_1,b_2)],\\
\chi_2 &:= \frac{1}{g-1}\Phi_4\left(\sum_{i=1}^{g}[[a_i,a_1],[b_i,a_1]]\right),\\
\chi_3 &:= \frac{1}{g-1}[\theta_1,\theta_2]
\end{aligned}$$

where

$$\theta_1 := \sum_{i=1}^{g}$$



$\in \mathfrak{h}_{g,1}^{\mathbb{Q}}(3)$, $\quad \theta_2 := \sum_{j=1}^{g}$



$\in \mathfrak{h}_{g,1}^{\mathbb{Q}}(1)$.

Note that $\chi_2, \chi_3 \in \operatorname{Ker}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(4) \to \mathfrak{h}_g^{\mathbb{Q}}(4))$. Indeed the commutative diagram

$$\begin{tikzcd}
 & \mathcal{L}_{g,1}^{\mathbb{Q}}(4) \arrow[r, "\Phi_4"] \arrow[d] & \mathfrak{h}_{g,1}^{\mathbb{Q}}(4) \arrow[d] & & \\
0 \arrow[r] & \mathcal{L}_g^{\mathbb{Q}}(4) \arrow[r, "\Psi_4"] & \mathfrak{h}_{g,*}^{\mathbb{Q}}(4) \arrow[r] & \mathfrak{h}_g^{\mathbb{Q}}(4) \arrow[r] & 0,
\end{tikzcd}$$

whose bottom row is exact, shows $\chi_2 \in \operatorname{Ker}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(4) \to \mathfrak{h}_g^{\mathbb{Q}}(4))$. Also, $\theta_2 \in [1] = \operatorname{Ker}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(1) \to \mathfrak{h}_g^{\mathbb{Q}}(1)) \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(1)$ and the fact that the map $\mathfrak{h}_{g,1}^{\mathbb{Q}} \to \mathfrak{h}_g^{\mathbb{Q}}$ preserves brackets show $\chi_3 \in \operatorname{Ker}(\mathfrak{h}_{g,1}^{\mathbb{Q}}(4)$

$\to \mathfrak{h}_g^{\mathbb{Q}}(4))$. By direct calculations, we have

	χ_1	χ_2	χ_3
$C_4^{(1,2)} \circ C_6^{(1,2)} \circ \iota_6$	0	$2a_1 \otimes a_1$	$(-8g-2)a_1 \otimes a_1$
$C_4^{(1,3)} \circ C_6^{(1,2)} \circ \iota_6$	$6a_1 \otimes a_1$	$(4g-2)a_1 \otimes a_1$	$(12g-2)a_1 \otimes a_1$
$C_4^{(1,2)} \circ C_6^{(1,3)} \circ \iota_6$	$-6a_1 \otimes a_1$	$-2a_1 \otimes a_1$	$-10a_1 \otimes a_1$

and from this we observe that χ_1, χ_2, χ_3 generate $3[2]$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(4)$.

Combining the above, we see that χ_1 survives in $\operatorname{Im} \tau_g^{\mathbb{Q}}(4) \subset \mathfrak{h}_g^{\mathbb{Q}}(4)$. □

5.3. **An upper bound of the kernel.** As in Section 4.3, we can find summands in $\wedge^2[2^2]$ which survive in $H^2(\mathcal{K}_g;\mathbb{Q})$ by showing the following.

Lemma 5.3. *For* $g \geq 4$, $\operatorname{Im}(\tau_g(2)_* : H_2(\mathcal{K}_g;\mathbb{Q}) \to H_2(\mathfrak{h}_g(2);\mathbb{Q}))$ *contains*

$$[431] + [32^21] + [321] + [21^2].$$

Proof. In the previous section, we have proved that

$$\operatorname{Im}(\tau_{g,1}(2)_* : H_2(\mathcal{K}_{g,1};\mathbb{Q}) \to H_2(\mathfrak{h}_{g,1}(2);\mathbb{Q})) \supset [431] + [32^21] + 2[321] + 2[21^2].$$

On the other hand, we have

$$\begin{aligned}
&\operatorname{Ker}(H_2(\mathfrak{h}_{g,1}(2);\mathbb{Q}) \to H_2(\mathfrak{h}_g(2);\mathbb{Q})) \\
=\ &\operatorname{Ker}(\wedge^2\mathfrak{h}_{g,1}^{\mathbb{Q}}(2) \to \wedge^2\mathfrak{h}_g^{\mathbb{Q}}(2)) \\
=\ &[2^2] \otimes [1^2] + \wedge^2[1^2] + [2^2] \otimes [0] + [1^2] \otimes [0] \\
=\ &([321] + [31] + [21^2] + [3^2] + [2^21^2] + [2^2] + [1^2]) + ([21^2] + [2]) + [2^2] + [1^2].
\end{aligned}$$

Hence at least $[431]+[32^21]$ is contained in $\operatorname{Im} \tau_g(2)_*$. $[321]$ is also contained in $\operatorname{Im} \tau_g(2)_*$, since the multiplicity of $[321]$ in $\wedge^2\mathfrak{h}_{g,1}^{\mathbb{Q}}(2)$ is 2 in which that in $\operatorname{Ker}(H_2(\mathfrak{h}_{g,1}(2);\mathbb{Q}) \to H_2(\mathfrak{h}_g(2);\mathbb{Q}))$ is 1. We now show that $[21^2]$ is certainly contained in $\operatorname{Im} \tau_g(2)_*$.

By Proposition 3.1, the projection $\wedge^2\mathfrak{h}_{g,1}^{\mathbb{Q}}(2) \to \wedge^2\mathfrak{h}_g^{\mathbb{Q}}(2)$ maps $T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_2,b_2,a_2,b_2) \in \operatorname{Im} \tau_{g,1}(2)_*$ to

$$\begin{aligned}
&\left\{T^H(a_1,b_1,a_1,b_1) - \frac{3}{g+1}\Phi_2(a_1 \wedge b_1) + \frac{3}{(2g+1)(g+1)}\Phi_2(\omega_0)\right\} \\
&\wedge \left\{T^H(a_2,b_2,a_2,b_2) - \frac{3}{g+1}\Phi_2(a_2 \wedge b_2) + \frac{3}{(2g+1)(g+1)}\Phi_2(\omega_0)\right\} \\
\\
=\ &T^H(a_1,b_1,a_1,b_1) \wedge T^H(a_2,b_2,a_2,b_2) \\
&-\frac{3}{g+1}\left\{(T^H(a_1,b_1,a_1,b_1) \wedge \Phi_2(a_2 \wedge b_2) + \Phi_2(a_1 \wedge b_1) \wedge T^H(a_2,b_2,a_2,b_2)\right\} \\
&+\frac{9}{(g+1)^2}\Phi_2(a_1 \wedge b_1) \wedge \Phi_2(a_2 \wedge b_2) \\
&+ (\text{the other summands}),
\end{aligned}$$

where the terms after the third are contained in $[2^2]\otimes[0]+[1^2]\otimes[0]$, and do not contribute to $[21^2]$. Then we have

$$
\begin{aligned}
&T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_2,b_2,a_2,b_2)\\
\overset{U_2U_3X_{1,2}X_{1,3}}{\longmapsto}\quad & 2T^H(a_1,a_2,a_1,a_3)\wedge T^H(a_2,b_2,a_2,b_2)\\
&-4T^H(a_1,b_1,a_1,a_3)\wedge T^H(a_1,a_2,a_2,b_2)\\
\overset{p_4^{(1,2,3)(4)}\circ C_6^{(3,5)}\circ C_8^{(1,7)}\circ\iota}{\longmapsto}\quad & 0,
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{g+1}\{T^H(a_1,b_1,a_1,b_1)\wedge\Phi_2(a_2\wedge b_2)+\Phi_2(a_1\wedge b_1)\wedge T^H(a_2,b_2,a_2,b_2)\}\\
=\quad & -\frac{3}{g+1}\sum_{i=1}^{g}\left\{\begin{array}{l} T^H(a_1,b_1,a_1,b_1)\wedge T^H(a_2,b_2,a_i,b_i)\\ +T^H(a_1,b_1,a_i,b_i)\wedge T^H(a_2,b_2,a_2,b_2)\end{array}\right\}\\
\overset{U_2U_3X_{1,2}X_{1,3}}{\longmapsto}\quad & -\frac{6}{g+1}\sum_{i=1}^{g}\left\{\begin{array}{l} T^H(a_1,a_2,a_1,a_3)\wedge T^H(a_2,b_2,a_i,b_i)\\ -T^H(a_1,b_1,a_1,a_3)\wedge T^H(a_1,a_2,a_i,b_i)\\ -T^H(a_1,a_3,a_i,b_i)\wedge T^H(a_1,a_2,a_2,b_2)\end{array}\right\}\\
\overset{p_4^{(1,2,3)(4)}\circ C_6^{(3,5)}\circ C_8^{(1,7)}\circ\iota}{\longmapsto}\quad & -\frac{144}{g+1}(a_1\wedge a_2\wedge a_3)\otimes a_1
\end{aligned}
$$

and

$$\frac{9}{(g+1)^2}\Phi_2(a_1\wedge b_1)\wedge\Phi_2(a_2\wedge b_2)\overset{X_{1,2}X_{1,3}}{\longmapsto}0.$$

Therefore $[21^2]$ is contained in $\operatorname{Im}\tau_g(2)_*$ for $g\geq 3$. □

From the above arguments, Theorem 5.1 follows.

Remark 5.4. Brendle-Farb [6] studied $H^2(\mathcal{K}_{g,1};\mathbb{Z})$ by using the Birman-Craggs-Johnson homomorphism and its integral lift by Morita [18], and showed that the rank of $H^2(\mathcal{K}_{g,1};\mathbb{Z})$ is at least $16g^4+O(g^3)$. From our computation, we can give a sharper estimate. Indeed the summand $[32^21]\subset\wedge^2[2^2]$ survives in $H^2(\mathcal{K}_{g,1};\mathbb{Q})$, $H^2(\mathcal{K}_{g,*};\mathbb{Q})$ and $H^2(\mathcal{K}_g;\mathbb{Q})$. The dimension of this summand is

$$\frac{1}{36}(g-3)(g-2)(g-1)(g+2)(2g-1)(2g+1)^2(2g+3).$$

However it seems that we should not compare these results by simply seeing their orders with respect to g, since the classes detected in [6] come from a different context from ours.

6. THE CASES OF $g=2,3$

We state the corresponding theorems for the cases of $g=2,3$. Note that we have already proved them in the argument of the previous sections.

6.1. **The case of $g = 2$.**

Lemma 6.1. *For $g = 2$, the irreducible decompositions of $\wedge^2[2^2]$, $[2^2] \otimes [1^2]$, $\wedge^2[1^2]$, $[2^2] \otimes [0]$ and $[1^2] \otimes [0]$ are given by the following table.*

	$\wedge^2[2^2]$	$[2^2] \otimes [1^2]$	$\wedge^2[1^2]$	$[2^2] \otimes [0]$	$[1^2] \otimes [0]$
$[42]$	1				
$[31]$		1			
$[2]$	1		1		
$[3^2]$		1			
$[2^2]$				1	
$[1^2]$		1			1

Table 2. *Irreducible decompositions of $Sp(4, \mathbb{Q})$-modules*

Theorem 6.2. *For $g = 2$, we have*

$$\mathrm{Ker}(\cup : \wedge^2([2^2] + [1^2] + [0]) \to H^2(\mathcal{K}_{2,1}; \mathbb{Q})) = [42] + [31] + 2[2],$$
$$\mathrm{Ker}(\cup : \wedge^2([2^2] + [1^2]) \to H^2(\mathcal{K}_{2,*}; \mathbb{Q})) = [42] + [31] + 2[2],$$
$$\mathrm{Ker}(\cup : \wedge^2[2^2] \to H^2(\mathcal{K}_2; \mathbb{Q})) = [42] + [2].$$

Since $\mathcal{K}_2$ is a free group [17], the third equality is trivial.

6.2. **The case of $g = 3$.**

Lemma 6.3. *For $g = 3$, the irreducible decompositions of $\wedge^2[2^2]$, $[2^2] \otimes [1^2]$, $\wedge^2[1^2]$, $[2^2] \otimes [0]$ and $[1^2] \otimes [0]$ are given by the following table.*

	$\wedge^2[2^2]$	$[2^2] \otimes [1^2]$	$\wedge^2[1^2]$	$[2^2] \otimes [0]$	$[1^2] \otimes [0]$
$[431]$	1				
$[42]$	1				
$[321]$	1	1			
$[31]$	1	1			
$[2^3]$	1				
$[21^2]$	1	1	1		
$[2]$	1		1		
$[3^2]$		1			
$[2^2]$		1		1	
$[1^2]$		1			1

Table 3. *Irreducible decompositions of $Sp(6, \mathbb{Q})$-modules*

Theorem 6.4. *For $g = 3$, we have*

$$\mathrm{Ker}(\cup : \wedge^2([2^2] + [1^2] + [0]) \to H^2(\mathcal{K}_{3,1}; \mathbb{Q})) = [42] + 2[31] + [2^3] + [21^2] + 2[2],$$
$$\mathrm{Ker}(\cup : \wedge^2([2^2] + [1^2]) \to H^2(\mathcal{K}_{3,*}; \mathbb{Q})) = [42] + 2[31] + [2^3] + [21^2] + 2[2],$$
$$\mathrm{Ker}(\cup : \wedge^2[2^2] \to H^2(\mathcal{K}_3; \mathbb{Q})) = [42] + [31] + [2^3] + [2].$$

7. Appendix

The following is a simple MATHEMATICA program which helps us to calculate the inclusion $\iota_k : \mathcal{L}_{g,1}^{\mathbb{Q}}(k) \hookrightarrow H_{\mathbb{Q}}^{\otimes k}$, the projections p_k^{σ} and the contractions $C_k^{(i,j)}$.

```
tens[]:=1

tens[x___,c_ y_,z___]:=
  c*tens[x, y,z]/;NumberQ[c]

tens[x___,c_,z___]:=
  c*tens[x,z]/;NumberQ[c]

tens[x___,c_ +y_,z___]:=
  tens[x, c,z]+tens[x,y,z]

SetAttributes[tens,Flat]

brac[x_,y_]:=tens[x,y]-tens[y,x]

tree[a_,b_,c_,d_,e_,f_]:=
  Expand[tens[a,brac[brac[brac[brac[f,e],d],c],b]]-
      tens[b,brac[brac[brac[brac[f,e],d],c],a]]-
      tens[c,brac[brac[brac[f,e],d],brac[b,a]]]-
      tens[d,brac[brac[brac[b,a],c],brac[f,e]]]+
      tens[e,brac[brac[brac[brac[b,a],c],d],f]]-
      tens[f,brac[brac[brac[brac[b,a],c],d],e]]]

Ht[a_,b_,c_,d_]:=
  Expand[tens[a,brac[b,brac[c,d]]]+tens[b,brac[brac[c,d],a]]+
      tens[c,brac[d,brac[a,b]]]+tens[d,brac[brac[a,b],c]]]

genusNo[x_]:=
  If[StringTake[ToString[x],{1}]=="a",
    ToExpression[
      StringDrop[ToString[x],1]],-ToExpression[StringDrop[ToString[x],1]]]

contract[a_,b_]:=
  If[genusNo[a]+genusNo[b]\[Equal]0,
    If[genusNo[a]>0,1,-1],0]

basisRecover[x_]:=
  If[x>0,ToExpression[ToString[SequenceForm["a",x]]],
    ToExpression[ToString[SequenceForm["b",-x]]]]

Wedge[]:=1

Wedge[x___,c_ y_,z___]:=
  c*Wedge[x, y,z]/;NumberQ[c]

Wedge[x___,c_,z___]:=
  c*Wedge[x,z]/;NumberQ[c]

Wedge[x___,c_ +y_,z___]:=
  Wedge[x, c,z]+Wedge[x,y,z]
```

```
Wedge[x___,c_,y___,c_,z___]:=0

Wedge[x___,c_,y___,d_,z___]:=
  -Wedge[x,d,y,c,z]/;compare[genusNo[c],genusNo[d]]\[Equal]1

SetAttributes[Wedge,Flat]

SetAttributes[genusNo,Listable]

compare[x_,y_]:=
  Which[x\[Equal]1/3,0,
    y\[Equal]1/3,0,
    Abs[x]>Abs[y],1,
    Abs[x]<Abs[y],0,
    x<y,1]
```

In this program, `tens` means $\otimes$, `Wedge` means $\wedge$, the function `tree[a_,b_,c_,d_,e_,f_]` makes $T(a,b,c,d,e,f)$ in $H_{\mathbb{Q}}^{\otimes 6}$, `Ht[a_,b_,c_,d_]` makes $T^H(a,b,c,d)$ in $H_{\mathbb{Q}}^{\otimes 4}$. The function `brac[x_,y_]` is the bracket operation. We now give a sample computation using this program below. We calculate

$$p_4^{(1,2)(3,4)} \circ C_6^{(1,2)} \circ C_8^{(1,2)} \circ \iota(4T^H(a_1,a_2,a_1,a_2) \wedge T^H(a_3,b_3,a_3,b_3))$$

of $[2^2]$ in Section 4.3. We first input the above program.

```
In[21]:=
xsi1=4brac[Ht[a1,a2,a1,a2],Ht[a3,b3,a3,b3]];

In[22]:=
ClearAttributes[tens,Flat];
xsi1/.tens[a_,b_,c_,d_,e_,f_,g_,h_]:>
    tens[contract[a,b],contract[c,d],Wedge[e,f],Wedge[g,h]]

SetAttributes[tens,Flat]

Out[23]=
-576 tens[a1\[Wedge]a2,a1\[Wedge]a2]
```

The output is $-576(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$.

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